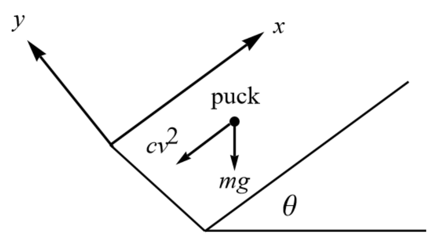
Problem 2.27

I kick a puck of mass m up an incline (angle of slope = θ) with initial speed v_o. There is no friction between the puck and the incline, but there is air resistance with magnitude $f(v) = cv^2$. Write down and solve Newton's second law for the puck's velocity as a function of t on the upward journey. How long does the upward journey last?

Solution

Draw a free-body diagram for a puck moving up an incline in a medium with quadratic air resistance.



Apply Newton's second law in the x-direction.

$$\sum F_x = ma_x$$

Let $v_x = v$ to simplify the notation.

$$-mg\sin\theta - cv^2 = m\frac{dv}{dt}$$

$$-c\left(\frac{mg}{c}\sin\theta + v^2\right) = m\frac{dv}{dt}$$

Solve this differential equation by separating variables.

$$-\frac{c}{m}\,dt = \frac{dv}{\frac{mg}{c}\sin\theta + v^2}$$

Integrate both sides definitely, assuming that at $t = 0$ the velocity is v_o.

$$\int_0^t -\frac{c}{m}\,dt' = \int_{v_o}^{v} \frac{dv'}{\frac{mg}{c}\sin\theta + v'^2}$$

$$-\frac{c}{m}t = \int_{v_o}^{v} \frac{dv'}{v'^2 + \frac{mg}{c}\sin\theta}$$

Make the following substitution in the remaining integral.

$$v' = \sqrt{\frac{mg}{c}\sin\theta}\tan\alpha$$

$$dv' = \sqrt{\frac{mg}{c}\sin\theta}\sec^2\alpha\,d\alpha$$

Consequently,

$$-\frac{c}{m}t = \int_{\tan^{-1}\left(\frac{v_o}{\sqrt{\frac{mg}{c}\sin\theta}}\right)}^{\tan^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}\sin\theta}}\right)} \frac{\sqrt{\frac{mg}{c}\sin\theta}\sec^2\alpha\,d\alpha}{\frac{mg}{c}\sin\theta\,(\tan^2\alpha+1)}$$

$$= \int_{\tan^{-1}\left(\frac{v_o}{\sqrt{\frac{mg}{c}\sin\theta}}\right)}^{\tan^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}\sin\theta}}\right)} \frac{\sec^2\alpha\,d\alpha}{\sqrt{\frac{mg}{c}\sin\theta}\sec^2\alpha}$$

$$= \frac{1}{\sqrt{\frac{mg}{c}\sin\theta}} \int_{\tan^{-1}\left(\frac{v_o}{\sqrt{\frac{mg}{c}\sin\theta}}\right)}^{\tan^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}\sin\theta}}\right)} d\alpha$$

$$= \frac{1}{\sqrt{\frac{mg}{c}\sin\theta}}\left[\tan^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}\sin\theta}}\right) - \tan^{-1}\left(\frac{v_o}{\sqrt{\frac{mg}{c}\sin\theta}}\right)\right]. \tag{1}$$

In order to find how long it takes for the puck to come to rest, set $v = 0$ and solve this equation for $t = t_{\text{rest}}$.

$$-\frac{c}{m}t_{\text{rest}} = \frac{1}{\sqrt{\frac{mg}{c}\sin\theta}}\left[\tan^{-1}\left(\frac{0}{\sqrt{\frac{mg}{c}\sin\theta}}\right) - \tan^{-1}\left(\frac{v_o}{\sqrt{\frac{mg}{c}\sin\theta}}\right)\right]$$

$$-\frac{c}{m}t_{\text{rest}} = -\sqrt{\frac{c}{mg\sin\theta}}\tan^{-1}\left(\frac{v_o}{\sqrt{\frac{mg}{c}\sin\theta}}\right)$$

Therefore,

$$\boxed{t_{\text{rest}} = \sqrt{\frac{m}{cg\sin\theta}}\tan^{-1}\left(v_o\sqrt{\frac{c}{mg\sin\theta}}\right).}$$

Now solve equation (1) for v.

$$-\frac{c}{m}t\sqrt{\frac{mg}{c}\sin\theta} = \tan^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}\sin\theta}}\right) - \tan^{-1}\left(\frac{v_o}{\sqrt{\frac{mg}{c}\sin\theta}}\right)$$

$$\tan^{-1}\left(\frac{v_o}{\sqrt{\frac{mg}{c}\sin\theta}}\right) - \frac{c}{m}t\sqrt{\frac{mg}{c}\sin\theta} = \tan^{-1}\left(\frac{v}{\sqrt{\frac{mg}{c}\sin\theta}}\right)$$

$$\tan\left[\tan^{-1}\left(\frac{v_o}{\sqrt{\frac{mg}{c}\sin\theta}}\right) - \frac{c}{m}t\sqrt{\frac{mg}{c}\sin\theta}\right] = \frac{v}{\sqrt{\frac{mg}{c}\sin\theta}}$$

As a result,

$$v(t) = \sqrt{\frac{mg}{c}\sin\theta}\tan\left[\tan^{-1}\left(v_{\text{o}}\sqrt{\frac{c}{mg\sin\theta}}\right) - t\sqrt{\frac{cg}{m}\sin\theta}\right].$$